Problem 1.45

Prove that if $\mathbf{v}(t)$ is any vector that depends on time (for example the velocity of a moving particle) but which has *constant magnitude*, then $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$. Prove the converse that if $\dot{\mathbf{v}}(t)$ is orthogonal to $\mathbf{v}(t)$, then $|\mathbf{v}(t)|$ is constant. [*Hint:* Consider the derivative of $\mathbf{v}^2$.] This is a very handy result. It explains why, in two-dimensional polars, $d\hat{\mathbf{r}}/dt$ has to be in the direction of $\hat{\boldsymbol{\phi}}$ and vice versa. It also shows that the speed of a charged particle in a magnetic field is constant, since the acceleration is perpendicular to the velocity.

Solution

Suppose there's a time-dependent vector,

$$\mathbf{v} = \mathbf{v}(t),$$

which has a constant magnitude,

$$|\mathbf{v}| = \lambda.$$

Consider the dot product of $\mathbf{v}$ with itself.

$$\begin{aligned} \mathbf{v}^2 &= \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{v}||\mathbf{v}| \cos 0 \\ &= (\lambda)(\lambda)(1) \\ &= \lambda^2 \end{aligned}$$

Take the derivative of both sides with respect to time.

$$\begin{aligned} \frac{d}{dt}(\lambda^2) &= \frac{d}{dt}\mathbf{v}^2 \\ 0 &= \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) \\ &= \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \\ &= \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \\ &= 2\left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right) \end{aligned}$$

Divide both sides by 2.

$$0 = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt}$$

Therefore, $\mathbf{v}$ is orthogonal to $\dot{\mathbf{v}} = d\mathbf{v}/dt$.

Suppose now that there's a time-dependent vector,

$$\mathbf{v} = \mathbf{v}(t),$$

such that $\mathbf{v}$ is orthogonal to $\dot{\mathbf{v}} = d\mathbf{v}/dt$.

$$\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 0$$

Multiply both sides by 2.

$$\begin{aligned} 0 &= 2\left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right) \\ &= \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \\ &= \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \\ &= \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) \end{aligned}$$

Integrate both sides with respect to time.

$$\begin{aligned} C &= \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{v}||\mathbf{v}|\cos 0 \\ &= |\mathbf{v}|^2 \end{aligned}$$

Take the square root of both sides.

$$|\mathbf{v}| = C^{1/2}$$

Therefore, $|\mathbf{v}|$ is constant.